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SOME REPRESENTATIONS OF THE SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF TWO-DIMENSIONAL STATIC THERMOELASTICITY*

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Exact solutions are obtained for special boundary value problems of linear thermoelasticity of a homogeneous anisotropic cylindrical body in the case when force and displacement in definite directions are given on the cross-sectional contour. These directions can vary consistently along the contour. The exact solutions of the special problems are used as the representations of solutions of other boundary value problems and to reduce a boundary value problem to the solution of a certain one-dimensional singular integral equation. The integral equation for an orthotropic strip is obtained as an illustration.

1. Let us consider a homogeneous anisotropic body in the shape of a cylinder of arbitrary cross-section on which force and temperature loads act without altering the cylindrical surface in the generatrix direction. We introduce a rectangular coordinate system such that two axes would be in the plane of the cross-section while the third would be directed along the cylinder generatrix. Then, the given forces p_i $(i = 1, 2, 3)$ distributed along the boundary surface, and the relative temperature ϑ will be functions of just the first two coordinates, i.e., $p_i = p_i(x_1, x_2)$ and $\vartheta = \vartheta(x_1, x_2)$. It is assumed that the forces p_i satisfy the equilibrium conditions on the boundary surface.

We assume the stresses corresponding to the loads p_{ij} $(i, j = 1, 2, 3)$ to be two-dimensional also. They then satisfy equilibrium equations of the form

$$\sum_{j=1}^{2} \partial_j p_{ij} = 0, \quad p_{ji} = p_{ij} \tag{1.1}$$

Here ∂_j is the partial differentiation operator with respect to x_j (mass forces are not taken into account, they can be taken into account by using singular solutions for an unbounded medium).

We write the governing relations for a body with rectilinear anisotropy in a form permitted with respect to the strains

$$\varepsilon_{ij} = \sum_{k,l=1}^{3} \alpha_{ijkl} p_{kl} + \delta_{ij}\alpha_i\vartheta, \quad i, j = 1, 2, 3 \tag{1.2}$$

Here the coefficients α_{ijkl} satisfy the symmetry relationships $\alpha_{ijlk} = \alpha_{ijkl}$, $\alpha_{jikl} = \alpha_{ijkl}$, α_i are the coefficients of linear expansion.

Let us note that a symmetry of the form $\alpha_{klij} = \alpha_{ijkl}$ is not assumed here since existing experimental values of the coefficients often do not satisfy this condition.

Since p_{33} is not in (1.1), we also eliminate it from the relationship (1.2). We will consequently have the following governing relationships

$$\varepsilon_{ij} = \sum_{k,l=1}^{3} a_{ijkl} p_{kl} + a_{ij}\vartheta + a_{ij0}\varepsilon_{33} \tag{1.3}$$

$$a_{ijkl} = \alpha_{ijkl} - a_{ij0}\alpha_{33kl}, \quad a_{ij} = \delta_{ij}\alpha_i - a_{ij0}\alpha_3, \quad a_{ij0} = \alpha_{ij33}/\alpha_{3333} \tag{1.4}$$

It follows from (1.4) that $a_{ij33} = a_{33kl} = a_{33} = 0$, and from (1.3) that the ε_{ij} depend also on only x_1, x_2, then the general strain compatibility conditions /1/ reduce to relationships of the form

$$\partial^2{}_2\varepsilon_{11} + \partial^2{}_1\varepsilon_{22} = 2\partial_1\partial_2\varepsilon_{12}, \quad \partial_2\varepsilon_{31} - \partial_1\varepsilon_{23} = c, \quad \varepsilon_{33} = c_0 + c_1x_1 + c_2x_2, \quad c,\ c_i = \text{const} \quad (i = 0, 1, 2) \tag{1.5}$$

Let us extract stress and temperature elements from their fields, namely of the form

$$p_{11}^* = p_{22}^* = p_{12}^* = 0, \quad p_{i33}^* = c_{i0} + c_{i1}x_1 + c_{i2}x_2, \quad i = 1, 2, 3, \quad \vartheta^* = a_0 + a_1x_1 + a_2x_2$$

For $c_{22} = -c_{11}$ the first of the fields satisfies the equilibrium equations (1.1) identically. The displacements u_i^* corresponding to these fields, that are found by integrating

*Prikl.Matem.Mekhan.,45,No.1,154-164,1981 the relationship $\partial_i u_j^* + \partial_j u_i^* = 2\varepsilon_{ij}^*$, will be polynomials of not more than the second order. We select the coefficients in the elementary field representations so as to satisfy the relations (1.5). Then for the remaining stress field $p_{ij}^0 = p_{ij} - p_{ij}^*$ and temperature field $\vartheta^0 = \vartheta - \vartheta^*$ the relations (1.5) will be homogeneous, i.e, we should set $c = c_i = 0$ therein. Therefore, we shall consider the fields p_{ij} considered below to be defined only to the accuracy of the elementary fields. Let us represent them by using the stress functions F and G:

$$p_{11} = \partial_2^2 F,\ p_{22} = \partial_1^2 F,\ p_{12} = -\partial_1\partial_2 F,\ p_{31} = \partial_2 G,\ p_{32} = -\partial_1 G \tag{1.6}$$

We obtain the following system of equations for the stress functions from (1.5) and (1.6), where $\varepsilon_{33} = 0$ and $c = 0$:

$$L_{11}F + 2L_{12}G = -M_1\vartheta,\ L_{21}F + 2L_{22}G = -M_2\vartheta \tag{1.7}$$

$$L_{11} = a_{1111}\partial_2^4 - 2(a_{1112} + a_{1211})\partial_2^3\partial_1 + (a_{1122} + a_{2211} + 4a_{1212})\partial_2^2\partial_1^2 - 2(a_{2212} + a_{1222})\partial_2\partial_1^3 + a_{2222}\partial_1^4$$

$$L_{22} = a_{3131}\partial_2^2 - (a_{3123} + a_{2331})\partial_2\partial_1 + a_{2323}\partial_1^2$$

$$L_{12} = a_{1131}\partial_2^3 - (a_{1123} + 2a_{1231})\partial_2^2\partial_1 + (a_{2231} + 2a_{1223})\partial_2\partial_1^2 - a_{2223}\partial_1^2$$

$$L_{21} = a_{3111}\partial_2^3 - (a_{2311} + 2a_{3112})\partial_2^2\partial_1 + (a_{3122} + 2a_{2312})\partial_2\partial_1^2 - a_{2322}\partial_1^3$$

$$M = a_{11}\partial_2^2 - 2a_{12}\partial_2\partial_1 + a_{22}\partial_1^2,\quad M_2 = a_{31}\partial_2 - a_{23}\partial_1$$

We note that $L_{21} = L_{12}$ if $a_{klij} = a_{ijkl}$.

We assume the temperature ϑ satisfies a heat conduction equation of the form

$$L_0\vartheta = 0,\ L_0 = k_{22}\partial_2^2 + 2k_{12}\partial_1\partial_2 + k_{11}\partial_1^2 \tag{1.8}$$

where the operator L_0 is strictly elliptic. This latter means that the characteristic polynomial $l_0 = k_{22}\gamma^2 + 2k_{12}\gamma + k_{22}$ has just complex roots $\gamma_0 = \alpha_0 + i\beta_0$ and $\bar\gamma_0$. Representation of the operator $L_0 = k_{22}(\partial_2 - \gamma_0\partial_1)(\partial_2 - \bar\gamma_0\partial_1)$ corresponds to expanding $l_0(\gamma)$ into factors. The kernel of each cofactor is analytic in the domain of a section of the function $T_1(z_0)$, $T_2(\bar z_0)$, $z_0 = x_1 + \gamma_0 x_2$, and according to the Boggio theorem /1/, the general (classical) solution of (1.8) has the form $\vartheta = T_1(z_0) + T_2(\bar z_0)$. Taking into account that ϑ is real, we obtain that $T_2 = \bar T$, $T_1 = T$, and we finally write the general solution in the form $\vartheta = 2\operatorname{Re} T(z_0)$, where $T(z_0)$ is an analytic function of a complex variable in the cross-sectional domain. We assume that $T(z_0)$ is also a certain particular solution of (1.8) that satisfies the given boundary condition.

According to the representation obtained for ϑ the system of equations (1.7) will have the form

$$L_{11}F + 2L_{12}G = -2\operatorname{Re}(m_1(\gamma_0)\,T''(z_0)),\ m_1 = a_{11}\gamma^2 - 2a_{12}\gamma + a_{22} \tag{1.9}$$

$$L_{21}F + 2L_{22}G = -2\operatorname{Re}(m_2(\gamma_0)\,T''(z_0)),\ m_2 = a_{31}\gamma - a_{23}$$

Here m_i are characteristic polynomials of the operator M_i.

Assuming the temperature field not to be reduced to an elementary field and hence $T''(z_0) \neq 0$, the particular solution of the system (1.9) is determined in the form

$$F = -2\operatorname{Re}(k_1(\gamma_0)\,H_0(z_0)),\quad G = -\operatorname{Re}(k_2(\gamma_0)\,H_0(z_0)),\quad H_0''(z_0) = T(z_0)$$

$$k_1 = (m_1 l_{22} - m_2 l_{12})/l,\ k_2 = (m_2 l_1 - m_1 l_{21})/l,\ l = l_{11}l_{22} - l_{12}l_{21}$$

Here l_{ij} are characteristic polynomials of the operators L_{ij}.

The general solution of the homogeneous system of equations corresponding to the system (1.9) is represented in the form $F = L_{22}H$, $G = -{}^1/_2 L_{21}H$, where the case when L_{21} and L_{22} can have a common nontrivial factor is eliminated here. Then H satisfies the equation

$$LH = 0,\quad L = L_{11}L_{22} - L_{12}L_{21}$$

The characteristic equation $l(\gamma) = 0$ has three complex roots $\gamma_n = \alpha_n + i\beta_n$, $n = 1, 2, 3$, and three conjugates (see /3/). We assume that all the roots are distinct. The case of equal roots is considered degenerate in the final solution of the problem. Expanding the polynomial $l(\gamma)$ into factors and representing the operator L analogously to the heat conduction operator L_0, we obtain the general solution in the form

$$H = 2\operatorname{Re}\sum_{n=1}^{3} H_n(z_n),\quad z_n = x_1 + \gamma_n x_2$$

Here $H_n(z_n)$ are analytic functions in the domain of definition of their arguments. The general solution of the system (1.9) has the form

$$F = 2\mathrm{Re}\left(\sum_{n=1}^{3} l_{22}(\gamma_n) E_n(z_n) - k_1(\gamma_0) H_0(z_0)\right), \quad G = -\mathrm{Re}\left(\sum_{n=1}^{3} l_{21}(\gamma_n) E_n(z_n) + k_2(\gamma_0) H_0(z_0)\right) \tag{1.10}$$

Here $E_n = H_n''(z_n)$ are analytic functions.

To determine E_n we obtain relationships on the boundary that correspond to the fundamental problems of elasticity theory. From the relationships for the stress functions on the section contour /3/

$$\partial_2 F = P_1, \quad -\partial_1 F = P_2, \quad G = P_3, \quad P_i = \pm \int p_i ds, \quad i = 1, 2, 3$$

where P_i are defined to the accuracy of nonessential constants selected so that P_i at a given point of the contour would take on the necessary values (the plus sign is taken for the outer contour of the section, and ds is the differential of an arc of the contour), the following conditions on the contour result for the functions $\Phi_n = E_n'$ $(j = 1, 2)$

$$2\mathrm{Re}\sum_{n=1}^{3} \gamma_n^{2-j} l_{22}(\gamma_n)\Phi_n(z_n) = (-1)^{j-1} T_j, \quad \mathrm{Re}\sum_{n=1}^{3} l_{21}(\gamma_n)\Phi_n(z_n) = -T_3 \tag{1.11}$$

$$T_j = P_j - 2\mathrm{Re}\left((-\gamma_0)^{2-j} k_1(\gamma_0) H_0'(z_0)\right), \quad T_3 = P_3 + \mathrm{Re}\left(k_2(\gamma_0) H_0'(z_0)\right) \tag{1.12}$$

For $T_j = 0$ the condition of equivalence of the temperature and force loads on the boundary follows from (1.12).

We now find the boundary conditions for the function $\Phi_n(z_n)$, that result from the boundary conditions for the displacements. By determining the stresses according to (1.6) and (1.10), we obtain

$$p_{ij} = 2\mathrm{Re}\left(\sum_{n=1}^{3} (-\gamma_n)^{4-i-j} l_{22}(\gamma_n)\Phi'(z_n) - (-\gamma_0)^{4-i-j} k_1(\gamma_0) T(z_0)\right) \tag{1.13}$$

$$p_{i3} = \mathrm{Re}\left(\sum_{n=1}^{3} (-\gamma_n)^{2-i} l_{21}(\gamma_n)\Phi_n'(z_n) + (-\gamma_0)^{2-i} k_2(\gamma_0) T(z_0)\right)$$

Integrating the equations $\partial_i u_j + \partial_j u_i = 2\varepsilon_{ij}$, having hence determined the strain from the relation (1.4) with (1.13) taken into account, we obtain for the displacements

$$u_i = 2\mathrm{Re}\left(\sum_{n=1}^{3} \gamma_n^{1-i}\varepsilon_{ii}^*(\gamma_n)\Phi_n(z_n) - \gamma_0^{1-i}\varepsilon_{ii0}^*(\gamma_0) H_0'(z_0)\right) \tag{1.14}$$

$$u_3 = 4\mathrm{Re}\left(\sum_{n=1}^{3} \varepsilon_{31}^*(\gamma_n)\Phi_n(z_n) - \varepsilon_{310}^*(\gamma_0) H_0'(z_0)\right)$$

$$\varepsilon_{ij}^* = (a_{ij11}\gamma^2 - 2a_{ij12}\gamma + a_{ij22})\, l_{22}(\gamma) - (a_{ij31}\gamma - a_{ij23})\, l_{21}(\gamma)$$

$$\varepsilon_{ij0}^* = a_{ij} - (a_{ij11}\gamma^2 - 2a_{ij12}\gamma + a_{ij22})\, k_1(\gamma) - (a_{ij31}\gamma - a_{ij23})\, k_2(\gamma)$$

Boundary conditions for $\Phi_n(z_n)$ result from (1.14).

2. To restore the analytic functions by means of their boundary values related by means of (1.11) and (1.14), we examine some special problems. The degree of definiteness of such problems is related to the nature of the material anisotropy. We consider only orthotropic material here, for which

$$a_{iikl} = 0,\ k \neq l;\ a_{ijkk} = 0,\ i \neq j;\ a_{ijkl} = 0,\ kl \neq ij$$

in the governing relationships (1.3).

In this case $L_{12} = L_{21} = 0$, and the system (1.7) decomposes into two independent equations $L_{11}F = -M_1\vartheta$ and $L_{22}G = 0$, which corresponds to separating the problem into bending with tension and torsion with longitudinal shear in the direction of the cylinder generatrix. This latter problem is not related to the temperature field. Let us examine just the first problem since the solution of the second is trivial.

The heat conduction operator has the form $L_0 = k_{22}\partial_2^2 + k_{11}\partial_1^2$, and therefore, $\gamma_0 = i\sqrt{k_{11}/k_{22}}$. The characteristic equation corresponding to the operator L_{11} is found in the form

$$l_{11}(\gamma) = a_{1111}\gamma^4 + (a_{1122} + a_{2211} + 4a_{1212})\gamma^2 + a_{2222} = 0 \tag{2.1}$$

and has two complex conjugate roots. Setting $l_{12} = l_{21} = 0$ in (1.10), we obtain the following representation for the stress function

$$F = 2\mathrm{Re}\left(\sum_{n=1}^{2} F_n(z_n) - m_1(\gamma_0) l_{11}^{-1}(\gamma_0) F_0(z_0)\right), \quad 2\mathrm{Re}\, F_0''(z_0) = \theta \tag{2.2}$$

Here $F_n(z_n)$ are analytic functions of their arguments, $z_n = x_1 + \gamma_n x_2$.

The boundary conditions for the functions $\Phi_n = F_n'$ will have the following form in terms of the force loads and temperature

$$2\mathrm{Re} \sum_{n=1}^{2} \gamma_n^{2-j} \Phi_n(z_n) = (-1)^{j-1} T_j, \qquad j = 1, 2 \tag{2.3}$$

$$T_j = P_j - 2\mathrm{Re}\,((-\gamma_0)^{2-j} m_1(\gamma_0) l_{11}^{-1}(\gamma_0) F_0'(z_0))$$

and in terms of the displacements and temperature

$$2\mathrm{Re} \sum_{n=1}^{2} \gamma_n^{1-j} \varepsilon_j(\gamma_n) \Phi_n(z_n) = U_j, \quad \varepsilon_j = a_{jj11}\gamma^2 + a_{jj22} \tag{2.4}$$

$$U_j = u_j + 2\mathrm{Re}\,(\gamma_0^{1-j}(a_{jj} - \varepsilon_j(\gamma_0) m_1(\gamma_0) l_{11}^{-1}(\gamma_0) F_0'(z_0))$$

Let us consider the following boundary value problem. Let T_θ and U_{θ^*} be the boundary "force" and "displacement" in the directions of the angles θ and θ^* to the x_1 axis, respectively. According to (2.3) and (2.4), we have on the boundary

$$T_\theta = T_1 \cos\theta + T_2 \sin\theta = 2\mathrm{Re} \sum_{n=1}^{2} t_n \Phi_n(z_n), \quad U_{\theta^*} = 2\mathrm{Re} \sum_{n=1}^{2} d_n \Phi_n(z_n) \tag{2.5}$$

$$t_n = \gamma_n \cos\theta - \sin\theta, \; d_n = \varepsilon_1(\gamma_n)\cos\theta^* + \gamma_n^{-1}\varepsilon_2(\gamma_n)\sin\theta^*$$

We introduce the auxiliary functions $\Psi_n = t_n \Phi_n(z_n)$. We represent the boundary conditions (2.5) in the form

$$2\mathrm{Re} \sum_{n=1}^{2} \Psi_n = T_\theta, \quad 2\mathrm{Re} \sum_{n=1}^{2} \varkappa_n \Psi_n = U_{\theta^*}, \quad \varkappa_n = \frac{d_n(\theta^*)}{t_n(\theta)} \tag{2.6}$$

Defining the angles θ and θ^*, so that the coefficients $\varkappa_n$ become real, i.e.,

$$\mathrm{Im}\,\varkappa_n(\theta, \theta^*) = 0, \; n = 1, 2 \tag{2.7}$$

Let us investigate the nontrivial solutions of these equations.

Depending on the sign of the determinant $d = (a_{1122} + a_{2211} + 4a_{1212})^2 - 4a_{1111}a_{2222}$ the characteristic equation (2.1) has roots of two kinds: if $d > 0$, then $\gamma_n = i\beta_n$, if $d < 0$, then $\gamma_n = \mp \alpha + i\beta$, $n = 1, 2$, and two conjugate roots (the case of equal roots is excluded).

For roots of the form $\gamma_n = i\beta_n$, condition (2.7) is possible in two cases: $\theta = 0, \theta^* = \pi/2$, or $\theta = \pi/2, \theta^* = 0$. In the first case, the force T_1 and the displacement U_2 are given on the body boundary, while in the second case the displacement U_1 and the force T_2 are given.

For roots of the form $\gamma_n = \mp\alpha + i\beta$, according to (2.7) the following formulas to define the angles are valid

$$\mathrm{tg}\,\theta = \pm \varepsilon (a_{2222}/a_{1111})^{1/4}, \quad \mathrm{tg}\,\theta^* = \mp \varepsilon (a_{1111}/a_{2222})^{1/4}$$

$$\varepsilon = ((\sqrt{a_{1111}a_{2222}} - a_{1122})/(\sqrt{a_{2222}a_{1111}} - a_{2211}))^{1/2}$$

For the values of θ and θ^* obtained, the roots are $\varkappa_2 \neq \varkappa_1$ in both cases, hence, the system (2.6) is solvable in a unique manner for $\mathrm{Re}\,\Psi_n$. Solving we obtain

$$\mathrm{Re}\,\Psi_1 = \frac{\varkappa_2 T_\theta - U_{\theta^*}}{2(\varkappa_2 - \varkappa_1)}, \quad \mathrm{Re}\,\Psi_2 = -\frac{\varkappa_1 T_\theta - U_{\theta^*}}{2(\varkappa_2 - \varkappa_1)} \tag{2.8}$$

The boundary value problem is therefore reduced to the restoration of the analytic functions in their domains of definition by means of their real parts on the boundary. Let $z = f(\zeta)$ be the mapping of some standard domain in the plane of the complex variable $\zeta = \xi + i\eta$ in a given domain of the complex variable $z = x + iy$ $(x = x_1, y = x_2)$. Any domain for which the Schwartz operator that restores a function analytic in a domain by means of its real part on the boundary is constructed effectively can be selected as standard. For definiteness, we assume that the cross-sectional domain is simply-connected and bounded by a piecewise-smooth curve, and the half-plane $\eta > 0$ is selected as standard domain.

The mappings

$$z_n = A_n z, \quad A_n = \begin{Vmatrix} 1 & \alpha_n \\ 0 & \beta_n \end{Vmatrix}, \quad n = 1, 2$$

(where $\alpha_n + i\beta_n = \gamma_n$ are the roots of the characteristic equation) carry the domain of the bar cross-section over into the corresponding domains of the planes of the complex variables $z_n = x_n + iy_n = x + \gamma_n y$, $n = 1, 2$. The mappings $z_n = A_n f(\zeta)$ are not conformal, hence, we consider the mappings $z_n = f_n(\zeta_n)$, $\zeta_n = \xi_n + i\eta_n$, $n = 1, 2$ simultaneously, which map the upper half-planes $\eta_n \geqslant 0$ conformally into images of the transverse section for the mappings $z_n = A_n z$. We then have the relationships $f_n(\zeta_n) = A_n f(\zeta)$, $n = 1, 2$ that connect the variables ζ_n and ζ.

Let us assume that T_θ and $U_{\theta*}$ are given as functions of z, and let us introduce the following real functions

$$\psi_n(\xi_n) = \frac{\varkappa_{3-n} T_\theta - U_{\theta*}(z)}{\varkappa_2 - \varkappa_1}, \quad z = A_n^{-1} f_n(\xi_n), \quad n = 1, 2 \tag{2.9}$$

According to (2.8), we obtain

$$2\,\mathrm{Re}\,\Psi_n = (-1)^{n-1}\psi_n(\xi_n), \; n = 1, 2$$

Here T_θ and $U_{\theta*}$ are given such that the functions ψ_n have a definite finite limit as $\xi_n \to \infty\ (\xi \to 0)$, and satisfy the Hölder condition on the whole axis. Applying the Schwartz operator for a half-plane, we obtain the following representations for the required functions:

$$\Psi_n(z_n) = \frac{(-1)^{n-1}}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_n(\tau_n)}{\tau_n - \zeta_n(z_n)} d\tau_n + ic, \quad \zeta_n(z_n) = f_n^{-1}(z_n)$$

The constant c is not essential for later, and will be omitted. If $\psi_n(\infty) \neq 0$, then $\psi_n(\tau_n)$ under the integral should be replaced by $\psi_n(\tau_n) - \psi_n(\infty)$.

The representations obtained for Ψ_n yield the exact solution of the boundary value problems in the case when the force T_θ and displacement $U_{\theta*}$ are given on the boundary of the section of a cylindrical body. In order to determine the connection between the obtained representations and the other boundary value problems, we assign the force and displacement in the direction of the angles ω and ω^* to the x axis. Then by using (2.5) in which θ should be replaced by ω, and having set $\Phi_n = t_n^{-1}\Psi_n$, where the relationship

$$\Psi_n = \frac{(-1)^{n-1}}{2}\left(\psi_n(\xi_n) + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\psi_n(\tau_n)}{\tau_n - \xi_n} d\tau_n\right)$$

is satisfied on the boundary according to the Sokhotskii—Plemelj formulas, we obtain

$$\begin{Vmatrix} T_\omega \\ U_{\omega*} \end{Vmatrix} = \sum_{n=1}^{2} (-1)^{n-1}\left(\mathrm{Re}\begin{Vmatrix} \sigma_n \\ \chi_n \end{Vmatrix}\psi_n(\xi_n) + \mathrm{Im}\begin{Vmatrix} \sigma_n \\ \chi_n \end{Vmatrix}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\psi_n(\tau_n)}{\tau_n - \xi_n} d\tau_n\right) \tag{2.10}$$

$$\sigma_n = t_n(\omega)/t_n(\theta), \quad \chi_n = d(\omega^*)/t_n(\theta)$$

The representations (10) result directly in integral equations for the solution of different boundary value problems. As an illustration, we consider the first boundary value problem of elasticity theory, when the forces T_1 and T_2, or equivalently, T_θ and T_ω are given on the boundary.

For roots of the characteristic equation of the form $\gamma_n = i\beta_n$ we have $\mathrm{Re}\,\sigma_2 = \mathrm{Re}\,\sigma_1 = \mathrm{Re}\,\sigma$. If the governing relations are symmetric, i.e., $a_{klij} = a_{ijkl}$, this is valid also for roots of the form $\gamma_n = \mp\alpha + i\beta$. We shall henceforth consider these cases only. From (2.9) and (2.10) we obtain an integral equation in $U^*(\xi) = U_{\theta*}(f(\xi))$ of the form

$$\sum_{n=1}^{2} \frac{(-1)^n \,\mathrm{Im}\,\sigma_n}{\pi} \int_{-\infty}^{\infty} \frac{U^*(\tau(\tau_n))}{\tau_n - \xi_n(\xi)} d\tau_n = T^*(\xi), \quad \xi_n(\xi) = f_n^{-1}(A_n f(\xi)) \tag{2.11}$$

$$T^*(\xi) = (\varkappa_2 - \varkappa_1)(T_\omega{}^*(\xi) - \mathrm{Re}\,\sigma\, T_\theta{}^*(\xi)) - \sum_{n=1}^{2} \frac{(-1)^n \varkappa_{3-n}\,\mathrm{Im}\,\sigma_n}{\pi} \times \int_{-\infty}^{\infty} \frac{T_\theta{}^*(\tau(\tau_n))}{\tau_n - \xi_n(\xi)} d\tau_n, \quad T^*_{\theta,\omega}(\xi) = T_{\theta,\omega}(f(\xi))$$

The equation obtained is a singular integral equation with shift, where questions on the existence of its solution are examined in /4,5/. Here the existence of a unique solution under appropriate assumptions will result from the following. We set $\omega = \theta^*$ and $\omega^* = \theta$ in (2.10) and we select two distinct pairs of values of θ and θ^*. Let us consider the case of the roots $\gamma_n = i\beta_n$. Setting $\theta = 0, \theta^* = \pi / 2$, we have from (2.9) and (2.10)

$$\left\|\begin{matrix} U_1^* \\ T_2^* \end{matrix}\right\| = \sum_{n=1}^{2} \frac{(-1)^{n-1}\beta_n}{\pi a_{2222}(\beta_2^2 - \beta_1^2)} \left\|\begin{matrix} \lambda_n \\ 1 \end{matrix}\right\| \int_{-\infty}^{\infty} \frac{\mu_{3-n}T_1^* - \beta_{3-n}^2 U_2^*}{\tau_n - \xi_n} d\tau_n \tag{2.12}$$

$$\lambda_n = a_{1111}\beta_n^2 - a_{1122}, \quad \mu_n = a_{2211}\beta_n^2 - a_{2222}$$

Setting $\theta = \pi / 2$ and $\theta^* = 0$, we obtain

$$\left\|\begin{matrix} T_1^* \\ U_2^* \end{matrix}\right\| = \sum_{n=1}^{2} \frac{(-1)^{n-1}\beta_n}{\pi a_{1111}(\beta_2^2 - \beta_1^2)} \left\|\begin{matrix} 1 \\ \mu_n/\beta_n^2 \end{matrix}\right\| \int_{-\infty}^{\infty} \frac{-\lambda_{3-n}T_2^* + U_1^*}{\tau_n - \xi_n} d\tau_n \tag{2.13}$$

There follows from the representation of (2.12) and (2.13) in operator form

$$\left\|\begin{matrix} U_1^* \\ T_2^* \end{matrix}\right\| = \mathbf{K}\left\|\begin{matrix} T_1^* \\ U_2^* \end{matrix}\right\|, \quad \left\|\begin{matrix} T_1^* \\ U_2^* \end{matrix}\right\| = \mathbf{N}\left\|\begin{matrix} U_1^* \\ T_2^* \end{matrix}\right\|$$

that the operators $\mathbf{K}$ and $\mathbf{N}$ are reciprocal. The existence of a unique solution of the appropriate integral equations in the class of Hölder-continuous functions results from the existence of the inverse operator. Meanwhile, the operators $\mathbf{K}$ and $\mathbf{N}$ are integral transforms with an inverse for any pair of Hölder-continuous functions on the whole number axis.

Let us now examine the case of equal roots for the characteristic equation, which we obtain by passing to the limit as $\beta_2 \to \beta_1$ in the relationships (2.12) and (2.13). We assume the function $\xi_2 = f_2^{-1}(A_2A_1^{-1}f_1(\xi_1))$, which maps the number axis into itself one-to-one, is smooth enough. Let $\gamma_n = i\beta_n$. Let us consider just the second relationship in (2.12), which we write in the form of an equation in U_2^*:

$$\beta_1\beta_2 \int_{-\infty}^{\infty} \frac{\beta_1 k(\xi_1, \tau_1) - \beta_2}{\beta_2 - \beta_1} \frac{U_2^*(\tau(\tau_1))}{\tau_1 - \xi_1} d\tau_1 = \pi a_{2222}(\beta_1 + \beta_2) T_2^*(\xi(\zeta_1)) + \int_{-\infty}^{\infty} \frac{\beta_2\mu_1 k(\xi_1, \tau_1) - \beta_1\mu_2}{\beta_2 - \beta_1} \frac{T_1^*(\tau(\tau_1))}{\tau_1 - \xi_1} d\tau_1 \tag{2.14}$$

$$k(\xi_1, \tau_1) = \tau_2'(\tau_1)(\tau_1 - \xi_1)/(\tau_2(\tau_1) - \xi_2(\xi_1))$$

Assuming

$$\beta_1 = \beta, \quad \beta_2 = \beta + \varepsilon, \quad \xi_2 = \varphi(\xi_1, \varepsilon) = \xi_1 + \varphi^{\cdot}(\xi_1)\varepsilon + o(\varepsilon)$$
$$\varphi^{\cdot}(\xi_1) = (\partial\varphi/\partial\varepsilon)_{\varepsilon=0}$$

and passing to the limit under the integral sign as $\varepsilon \to 0$, we obtain an integral equation for the case of equal roots in the form

$$\frac{\beta}{2\pi a_{2222}} \int_{-\infty}^{\infty} \frac{1 + \beta k^\circ(\xi_1, \tau_1)}{\tau_1 - \xi_1} U_2^*(\tau(\tau_1)) d\tau_1 = -T_2^*(\xi(\xi_1)) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\beta^{-1} - \lambda\beta - (1 + \lambda\beta^2) k^\circ(\xi_1, \tau_1)}{\tau_1 - \xi_1} T_1^*(\tau(\tau_1)) d\tau_1 \tag{2.15}$$

$$k^\circ(\xi_1, \tau_1) = \frac{\varphi^{\cdot}(\tau_1) - \varphi^{\cdot}(\xi_1)}{\tau_1 - \xi_1} - \frac{\partial\varphi^{\cdot}(\tau_1)}{\partial\tau_1}, \quad \lambda = -\frac{a_{2211}}{a_{2222}}$$

For an isotropic material $a_{2222} = (1 - \nu^2) / E$, $a_{2211} = -\nu(1 + \nu)/E$, $\lambda = \nu(1 - \nu)$, ν is the Poisson's ratio, E is the elastic modulus, the characteristic equation (2.1) has equal roots $\gamma_n = i$ such that $\beta = 1$, and $\tau_1 = \tau, \xi_1 = \xi$ should be put into (2.15).

Example. If $k(\xi_1, \tau_1) = 1$ in (2.14), and $k^\circ(\xi_1, \tau_1) = 0$ in (2.15), then these equations are solved exactly by applying the Cauchy inversion operator. The dependences $\xi_2 = a\xi_1 + b$, $\varphi^{\cdot}(\xi_1) = a\xi_1 + b$, $a, b = \text{const}$ correspond, respectively, to these cases, where a particular case of the former is used in /3/ to solve the anisotropic problem of deformation of a domain outside a parabola. The transformations $z_n = A_n z$ transfer a parabola into the parabolas, and the domains outside the parabolas are images of the half-plane under the mappings $z_n = \beta_n^2(\zeta_n^2 + i2q\zeta_n) - q^2$, where q is the parabola parameter. The relation between ζ_2 and ζ_1 on the half-plane boundary yields $\beta_2\xi_2 = \beta_1\xi_1$, which corresponds to the case under consideration. Let us present the solution for an isotropic medium

$$U_2^*(\xi)=\frac{(1-2\nu)(1+\nu)}{2E}T_1^*(\xi)+\frac{1-\nu^2}{\pi E}\int_{-\infty}^{\infty}\frac{T_2(\tau)}{\tau-\xi}d\tau$$

Equations (2.14) and (2.15) are obtained under the assumption of sufficient smoothness of the functions therein. Let us now extend these equations to the case of a more general form of the function $\xi_2=\varphi(\xi_1)$. We convert the integral equation (2.14) to a form where smoothness conditions are not formally present for the function mentioned. To do this we first obtain a formula to rearrange the singular integrals with special kernels.

For a real Hőlder-continuous function $u(\xi)$ on the whole number axis, and for a function $\varphi(\xi)$ mapping the number axis into itself one-to-one, where its derivative is not zero and is Hőlder-continuous, we have

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{\rho-\xi}\int_{-\infty}^{\infty}\frac{u(\tau)\,d\tau}{\tau-\varphi(\rho)}d\rho=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{\rho-\xi}\int_{-\infty}^{\infty}\frac{v(\rho,\sigma)\,d\sigma}{\sigma-\rho}d\rho\,,\qquad v(\rho,\sigma)=\frac{u(\varphi(\sigma))\,\varphi'(\sigma)(\sigma-\rho)}{\varphi(\sigma)-\varphi(\rho)}$$

The function v is here Hőlder-continuous. Applying the Bertrand—Poincaré commutation to the last integral, we obtain

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{\rho-\xi}\int_{-\infty}^{\infty}\frac{u(\tau)\,d\tau}{\tau-\varphi(\rho)}d\rho=-u(\varphi(\xi))+\frac{1}{\pi^2}\int_{-\infty}^{\infty}u(\tau)\int_{-\infty}^{\infty}\frac{d\rho}{(\rho-\xi)(\tau-\varphi(\rho))}d\tau \tag{2.16}$$

Here the inner integral on the right is understood in the principal value sense with respect to the point ξ and $\varphi^{-1}(\tau)$.

Now applying the operator S to (2.11) so that

$$Su(\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\rho)}{\rho-\xi}d\rho$$

taking into account that $S^2u=-u$, and assuming the conditions of (2.16) to be satisfied, we obtain the following integral equation

$$\operatorname{Im}(\sigma_1-\sigma_2)U^*(\xi)+\frac{\operatorname{Im}\sigma_2}{\pi}\int_{-\infty}^{\infty}k(\xi_1(\xi),\tau_2)U^*(\tau(\tau_2))\,d\tau_2=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{T^*(\rho(\rho_1))}{\rho_1-\xi_1(\xi)}d\rho_1 \tag{2.17}$$

$$k(\xi_1,\tau_2)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\rho_1}{(\rho_1-\xi_1)(\tau_2-\varphi(\rho_1))}$$

As an application of (2.17), we obtain the integral equation for bending of a strip. The mappings $z_n=A_nz$ transfer a strip of width h parallel to the x-axis into similarly located strips of width β_nh for any roots of the characteristic equation. For definiteness, we consider the case $\gamma_n=\mp\alpha+i\beta$, which holds for many kinds of wood /6/, for birch, say. The upper half-planes are mapped into strips by the functions $z_n=\pi^{-1}\beta_nh\ln\zeta_n$ so that we have on the boundary

$$\xi_1\geqslant 0,\ \xi_2=\xi_1;\ \xi_1\leqslant 0,\ \xi_2=c\xi_1,\ c=\exp(2\pi\alpha/\beta)$$

We obtain from (2.17) for $\xi_1\neq 0,\ \tau_2\neq 0$

$$k(\xi_1,\tau_2)=\frac{\ln|\tau_2|-\ln|\xi_1|}{\pi(\tau_2-\xi_1)}-\frac{\ln|\tau_2|-\ln|c\xi_1|}{\pi(\tau_2-c\xi_1)},\quad \tau_2\neq\xi_1,\ c\xi_1$$

We have for the remaining ξ_1 and τ_2

$$\xi_1>0,\quad k(\xi,c\xi_1)=\frac{1}{\pi\xi_1}\left(\frac{\ln c}{c-1}-\frac{1}{c}\right);\quad \xi_1<0,\quad k(\xi_1,c\xi_1)=\infty$$

$$\xi_1>0,\quad k(\xi_1,\xi_1)=\infty;\quad \xi_1<0,\quad k(\xi_1,\xi_1)=\frac{1}{\pi\xi_1}\left(1-\frac{\ln c}{c-1}\right)$$

$$k(0,\tau_2)=2\pi\alpha/\beta,\ k(\xi_1,0)=\infty$$

3. We now construct a more general representation for an orthotropic material in the case when the boundary conditions can change along the boundary, namely, we assume that θ and its conjugate angle θ^* are a function of the boundary points. To do this we assume that the

relationship (2.7) is satisfied just for some one n. In the case of the roots $\gamma_n = i\beta_n$, $n = 1, 2$ there hence follows a dependence between the angles of the form

$$\operatorname{ctg}\theta^* = \frac{a_{221} - a_{222}\beta_n^{-2}}{a_{111}\beta_n^{2} - a_{112}} \operatorname{tg}\theta$$

For difiniteness, we select $n = 2$. In place of (2.6), we then obtain

$$2\operatorname{Re}\Psi_1 + 2\operatorname{Re}\Psi_2 = T_\theta, \quad 2\operatorname{Re}(\varkappa_1\Psi_1) + 2\varkappa_2\operatorname{Re}\Psi_2 = U_{\theta^*} \tag{3.1}$$

Eliminating Ψ_2 we obtain $2\operatorname{Re}((\varkappa_2 - \varkappa_1)\Psi_1) = \varkappa_2 T_\theta - U_{\theta^*}$. Assuming that $\varkappa_2 - \varkappa_1$ depends on points of the boundary such that $(\varkappa_2 - \varkappa_1)\Psi_1$ is the boundary value of an analytic function, we analogously find

$$\Psi_1(z_1) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\psi_1(\tau_1)}{\tau_1 - \zeta_1(z_1)}\, d\tau_1 \tag{3.2}$$

Here the function ψ_1 is determined from (2.9), but is already not real. Determining $\operatorname{Re}\Psi_1$ on the boundary from (3.2) by using the Sokhotskii—Plemelj formula, we obtain from the first relationship in (3.1)

$$\Psi_2(z_2) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{T_\theta^*(\tau(\tau_2)) - 2\operatorname{Re}\Psi_1^*(\tau(\tau_2))}{\tau_2 - \zeta_2(z_2)}\, d\tau_2$$

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